

# Regularity and symmetries of nonholonomic systems

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## Abstract

Lagrangian systems with nonholonomic constraints may be considered as singular differential equations defined by some constraints and some multipliers. The geometry, solutions, symmetries and constants of motion of such equations are studied within the framework of linearly singular differential equations. Some examples are given; in particular the well-known singular lagrangian of the relativistic particle, which with the nonholonomic constraint  $v^2 = c^2$  yields a regular system.

*Key words: nonholonomic constraint, singular differential equation, symmetry, constant of motion*

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## 1 Introduction

The main goal of this paper is to study the nonholonomic mechanical systems within the framework of linearly singular differential equations.

Nonholonomic mechanical systems, i.e., mechanical systems with non-integrable kinematic constraints, have been discussed since the last years of nineteenth century. However, the geometric foundations for the theory were given in [VF 72]. Since then, several approaches have been taken to deal with the subject, for instance, a hamiltonian approach in [BS 93], a lagrangian approach in [LM 96], a more general Poisson framework in [Mar 98], or an approach based on a gauge independent formulation of lagrangian and hamiltonian mechanics in [MVB 02]. Symmetries of these systems, as well as reduction schemes derived from them, have also been considered in the literature, see [Koi 92, BKMM 96, KM 98, CL 99, Mar 03].

A lagrangian system with nonholonomic constraints may be considered, more generally, as a singular differential equation defined by some constraints and some multipliers:

$$\dot{x} = g(x) + \sum_{\mu} u^{\mu} h_{\mu}(x), \quad \phi_{\alpha}(x) = 0.$$

Such an equation can be described geometrically as a *linearly singular differential equation*, that is, a differential equation where the velocities are not isolated because of a linear factor multiplying them:

$$A(x)\dot{x} = b(x).$$

This is a special type of *implicit differential equation*. The idea of modelling mechanical systems as implicit differential equations is found in earlier papers by Tulczyjew [MT 78, MMT 95], and it has also been used to deal with nonholonomic constraints [Tul 86, ILMM 96].

Linearly singular differential equations were geometrically presented in [GP 91, GP 92]. This general framework includes for instance the presymplectic systems and the lagrangian formalism [GP 92], the higher order singular lagrangians and their “higher order differential equation” conditions [GPR 91], as well as many other systems that appear in technological applications. To solve the corresponding equation of motion a consistency algorithm can be performed. This algorithm is indeed a generalization of the presymplectic constraint algorithm [GNH 78].

We will see that a system with constraints and multipliers, and in particular any nonholonomic mechanical system, can be described as a linearly singular system. This implies that all the methods and results about these systems can be applied directly to nonholonomic systems.

More precisely, the combination of two operations that can be performed on linearly singular systems —restriction to a subsystem and projection to a quotient— can be applied to obtain what we call a generalized nonholonomic system. In particular, we discuss the regularity, consistency and equations of motion of these derived systems.

The symmetries of a linearly singular differential equation have been studied in [GP 02]. In this paper we consider the relation between the symmetries of a system with nonholonomic constraints and the symmetries of its original unconstrained system, both modelled on linearly singular differential equations. We also study their constants of motion.

The paper is organized as follows. In section 2 we give some definitions and results regarding linearly singular differential equations, their solutions and their symmetries. In section 3 we introduce generalized nonholonomic systems and discuss some of their properties. Symmetries and constants of motion of generalized nonholonomic systems are discussed in section 4. In section 5 we show how a lagrangian system with nonholonomic constraints can be described in terms of a generalized nonholonomic systems. The case of a relativistic particle is studied in section 6, where we see that a nonholonomic constraint can convert a singular lagrangian into a regular system. Two additional examples are studied in section 7. Finally, an appendix contains some auxiliary results formulated within the framework of linear algebra.

## 2 Previous results: linearly singular systems

In this section we recall some definitions and results from [GP 91, GP 92, GP 02].

Let  $M$  be a manifold. An *implicit differential equation* on  $M$  is defined by a submanifold  $D \subset TM$ . A path  $\xi: I \rightarrow M$  is a solution of this equation when

$$\dot{\xi}(I) \subset D. \tag{2.1}$$

In coordinates, if the submanifold  $D$  is described by some equations  $F = 0$  and the path  $\xi$  is represented by some functions  $x(t)$ , then the local expression of the implicit differential equation is  $F(x, \dot{x}) = 0$ .

We have a particular case when  $D = X(M)$ , with  $X$  a vector field on  $M$ . Then  $X$  defines an *explicit differential equation*, and  $\xi$  is a solution iff

$$\dot{\xi} = X \circ \xi. \quad (2.2)$$

Now the local expression is  $\dot{x} = f(x)$ .

A *linearly singular differential equation* on  $M$  is defined by a vector bundle  $\pi: F \rightarrow M$ , a vector bundle morphism  $A: TM \rightarrow F$ , and a section  $f: M \rightarrow F$  of  $\pi$ . A path  $\xi: I \rightarrow M$  is a solution when

$$A \circ \dot{\xi} = f \circ \xi, \quad (2.3)$$

whose local expression is  $A(x)\dot{x} = f(x)$ , with  $A(x)$  a (singular, in general) matrix.

We denote by  $(A: TM \rightarrow F, f)$  the linearly singular system. The following diagram shows all these data:

$$\begin{array}{ccc} & TM & \xrightarrow{A} F \\ & \uparrow \dot{\xi} & \uparrow \pi \\ I & \xrightarrow{\xi} M & \end{array} \quad \begin{array}{c} \downarrow \tau_M \\ \end{array} \quad \begin{array}{c} \nearrow f \end{array}$$

The associated implicit differential equation is

$$D = A^{-1}(f(M)) \subset TM. \quad (2.4)$$

We say that the linearly singular differential equation is *regular* when  $A$  is a vector bundle isomorphism. In this case, the associated explicit differential equation is given by the vector field  $X = A^{-1} \circ f$ .

The solutions of the system can be equivalently described as integral curves of vector fields. Let us remark that in general the solutions are restricted to a submanifold  $S \subset M$  because the equation (2.3) may not have solutions passing through every point  $x \in M$ . Therefore, the equation of motion can be written as an equation for a vector field  $X$  and a submanifold  $S$ :

$$\begin{cases} X \text{ tangent to } S \\ A \circ X \underset{S}{\simeq} f, \end{cases} \quad (2.5)$$

where the notation  $\underset{S}{\simeq}$  means equality at the points of  $S$ .

A recursive algorithm can be applied to find the solutions of a linearly singular differential equation. Its first step is to note that, in order that a solution passes through a point  $x \in M$ , it is necessary that

$$f(x) \in \text{Im } A_x, \quad (2.6)$$

so the solutions are necessarily contained in the primary constraint subset

$$M_1 = \{x \in M \mid f(x) \in \text{Im } A_x\}, \quad (2.7)$$

which will be assumed to be a closed submanifold. The tangency to  $M_1$  forces the initial system to be restricted to  $(A_1: TM_1 \rightarrow F_1, f_1)$ , where  $A_1 = A|_{TM_1}$ ,  $F_1 = F|_{M_1}$  and  $f_1 = f|_{M_1}$ . The algorithm follows recursively, and, under some regularity assumptions at each step, it ends with a final constraint submanifold  $S$  such that  $f(S) \subset \text{Im } A_S$ ; thus the system is consistent, and the equation

$$A_S \circ X = f_S \quad (2.8)$$

for a vector field  $X$  tangent to  $S$  has solutions. Given a particular solution  $X_\circ$ , the set of solutions of (2.5) is  $X_\circ + \text{Ker } A_S$ .

We finish this section by giving some definitions and results about symmetries. A *symmetry* of a linearly singular system  $(A: TM \rightarrow F, f)$  is a vector bundle automorphism  $(\varphi, \Phi)$  of  $\pi: F \rightarrow M$  such that

$$f = \Phi_*[f] := \Phi \circ f \circ \varphi^{-1}, \quad A = \Phi_*[A] := \Phi \circ A \circ (T\varphi)^{-1}. \quad (2.9)$$

An *infinitesimal symmetry* of a linearly singular system  $(A: TM \rightarrow F, f)$  is an infinitesimal automorphism  $(V, W)$  of the vector bundle  $\pi: F \rightarrow M$  such that its flow  $(F_V^\varepsilon, F_W^\varepsilon)$  is constituted by local symmetries of the linearly singular differential equation. The last property is equivalent to the following conditions:

$$Tf \circ V = W \circ f, \quad TA \circ V^T = W \circ A, \quad (2.10)$$

which are the infinitesimal version of (2.9).

### 3 Generalized nonholonomic systems

#### The geometric setting

Among the various operations that can be performed with a linearly singular system  $(B: TN \rightarrow G, g)$ , we are especially interested in the subsystem defined on a submanifold  $j: M \hookrightarrow N$ , and the projection  $p: G \rightarrow G/G'$  to a quotient with respect to a vector subbundle  $G' \subset G$ :

$$\begin{array}{ccc} TN & \xrightarrow{B} & G \\ \downarrow & \nearrow g & \\ N & & \end{array} \quad \begin{array}{ccc} TM & \xrightarrow{B|_{TM}} & G|_M \\ \downarrow & \nearrow g|_M & \\ M & & \end{array} \quad \begin{array}{ccc} TN & \xrightarrow{p \circ B} & G/G' \\ \downarrow & \nearrow p \circ g & \\ N & & \end{array}$$

Suppose that the original system admits solutions  $Y$  on a submanifold  $N_f \subset N$ . Then the subsystem on  $M$  has solutions on the submanifolds of  $M \cap N_f$  over which a solution  $Y$  of the initial system is tangent. On the other hand, the quotient system has, in general, more solutions than the initial system: if  $Z$  is any vector field on  $N$  tangent to  $N_f$  with values in  $B^{-1}(G')$  then  $Y + Z$  is a solution of the quotient system on  $N_f$ ; there may also exist solutions defined on a submanifold greater than  $N_f$ .

It is well known that the dynamics of systems with nonholonomic constraints is a mixture of both constructions: the presence of some constraints, combined with a certain degree of

arbitrariness expressed through some multipliers. This combination may result advantageous: though in general  $Y$  is not tangent to the submanifold  $M$ , it may happen that for some vector fields  $Z$  in  $B^{-1}(G')$  one has solutions  $Y + Z$  tangent to  $M$ , or at least to a “big” submanifold of  $M$ .

In this paper we will call a *generalized nonholonomic system* the linearly singular system  $(A: TM \rightarrow F, f)$  defined from  $(B: TN \rightarrow G, g)$  by a *constraint submanifold*  $M \subset N$  and a *subbundle of constraint forces*  $G' \subset G|_M$  as follows:

- $F = (G|_M)/G'$ ,
- $A = p \circ B|_M \circ \mathring{T}j$ , and
- $f = p \circ g|_M$ ,

where  $p: G|_M \rightarrow (G|_M)/G'$  is the projection to the quotient, and  $\mathring{T}j$  denotes the tangent map of  $j$  with the image restricted to  $M$ . All this is shown in the following diagram:

$$\begin{array}{ccccccc}
 & & & \xrightarrow{\quad A \quad} & & & \\
 TM & \xrightarrow{\quad \mathring{T}j \quad} & TN|_M & \xrightarrow{\quad B|_M \quad} & G|_M & \xrightarrow{\quad p \quad} & F = (G|_M)/G' \\
 \downarrow & & \downarrow & \nearrow g|_M & & \nearrow f & \\
 M & \xlongequal{\quad} & M & & & & 
 \end{array}$$

## Regularity and consistency

Before discussing the equations of motion, we want to study some general properties of the generalized nonholonomic system  $(A: TM \rightarrow F, f)$ , namely, whether  $A$  is surjective (we will also say that the system is surjective) or bijective (the system is regular), or the equation  $A \circ X = f$  is everywhere consistent.

Let us denote

$$H = B^{-1}(G') \subset TN|_M,$$

which is a vector subbundle whenever the morphism  $B$  has constant rank.

**Proposition 1** *With the preceding notations, the generalized nonholonomic system is surjective iff*

$$B(TM) + G' = G|_M.$$

*Assuming that the original system is surjective, the nonholonomic system is surjective iff*

$$TM + H = TN|_M,$$

*and it is regular iff in addition*

$$TM \cap H = \{0\}.$$

*Proof.* We want to decide whether  $A = p \circ B|_M \circ \overset{\circ}{T}j$  (the composition of an inclusion, a morphism and a projection) is surjective or injective, and this is given by lemma 1 in the appendix. ■

The preceding result could be refined also in the case where  $B$  is injective, but this does not seem so interesting. As an immediate consequence, we have:

**Corollary 1** *Suppose that the original system is surjective (or, more particularly, regular). Then the generalized nonholonomic system is regular iff*

$$TN|_M = TM \oplus H.$$

■

These relations can be given a more concrete form in terms of constraints and frames. Consider a local basis  $(\Gamma_\mu)_{1 \leq \mu \leq m_\circ}$  of sections for the subbundle  $H \subset TN|_M$  (they are vector fields in  $N$ , but defined only on  $M$ ). Consider also a set of  $a_\circ$  constraints  $\phi^\alpha$ , linearly independent at each point, that locally define the submanifold  $M \subset N$ . Finally, consider the matrix

$$D_\mu^\alpha = \langle d\phi^\alpha|_M, \Gamma_\mu \rangle = \Gamma_\mu \cdot \phi^\alpha, \quad (3.1)$$

whose elements are functions on  $M$ .

**Proposition 2** *With the preceding notations,*

1.  $TM \cap H = 0$  iff  $\text{rank}(D_\mu^\alpha) = m_\circ$ .
2.  $TM + H = TN|_M$  iff  $\text{rank}(D_\mu^\alpha) = a_\circ$ .
3.  $TM \oplus H = TN|_M$  iff  $(D_\mu^\alpha)$  is a square invertible matrix.

*Proof.* It is a consequence of lemma 3 in the appendix, since the  $d\phi^\alpha|_M$  constitute a basis for the annihilator of  $TM$  in  $(TN|_M)^*$ . ■

The connection of such a matrix with the notion of regularity and consistency of a constrained system was already noted in [CR 93, LM 96].

## Equations of motion

From the definition of the generalized nonholonomic system  $(A: TM \rightarrow F, f)$ , it is clear that a path  $\xi: I \rightarrow N$  is a solution of the equation of motion iff it is contained in  $M$  and

$$B \circ \dot{\xi} - g \circ \xi \in G'. \quad (3.2)$$

If some sections  $\Delta_\nu$  constitute a frame for  $G'$ , then this equation can be written as

$$B \circ \dot{\xi} = g \circ \xi + \sum_\nu v^\nu \Delta_\nu \circ \xi, \quad (3.3)$$

for some multipliers  $v^\nu(t)$ .

In the same way, for a submanifold  $S \subset M$  and a vector field  $X$  on  $M$  tangent to  $S$ , the equation of motion  $A \circ X \underset{S}{\simeq} f$  can be written as

$$B \circ X - g \in \underset{S}{G}', \quad (3.4)$$

where the equation must only hold on the points of  $S$ . This equation may be also written as

$$B \circ X \underset{S}{\simeq} g + \sum_{\nu} v^{\nu} \Delta_{\nu}, \quad (3.5)$$

for some multipliers  $v^{\nu}(x)$ .

Of course, we can apply the constraint algorithm to find the solutions of this linearly singular system. However, there is an alternative way to solve the problem when the original problem is regular, or at least consistent. Under this hypothesis, let  $Y$  be a vector field on  $N$ , solution of the equation of motion of the linearly singular system  $(B: TN \rightarrow G, g)$ :

$$B \circ Y = g.$$

(For most applications the original system is regular, and then the unique solution of this equation is the vector field  $Y = B^{-1} \circ g$ .)

Using  $Y$ , the equations of motion become

$$\dot{\xi} - Y \circ \xi \in H \quad (3.6)$$

for a path  $\xi$  in  $M$ , and

$$X - Y \underset{S}{\subseteq} H, \quad (3.7)$$

for a vector field  $X$  on  $M$  that should be tangent to  $S$ .

These equations can be expressed in a more concrete form in terms of the local basis  $(\Gamma_{\mu})$  of sections for the subbundle  $H \subset TN|_M$ :

$$\dot{\xi} = Y \circ \xi + \sum_{\mu} u^{\mu} \Gamma_{\mu} \circ \xi, \quad (3.8)$$

for some functions  $u^{\mu}(t)$ , and

$$X \underset{S}{\simeq} Y + \sum_{\mu} u^{\mu} \Gamma_{\mu}, \quad (3.9)$$

for some functions  $u^{\mu}$  on  $M$ .

Let us examine whether this last equation has solutions. The requirement for  $X$  of being tangent to  $M$  is  $X \cdot \phi^{\alpha} \underset{M}{\simeq} 0$ , which reads

$$\sum_{\mu} D_{\mu}^{\alpha} u^{\mu} + Y \cdot \phi^{\alpha} \underset{M}{\simeq} 0, \quad (3.10)$$

where  $(D_{\mu}^{\alpha})$  is the matrix defined by (3.1). From this it is clear that the generalized nonholonomic system is regular iff the matrix  $(D_{\mu}^{\alpha})$  is invertible on  $M$ , and in this case the equation (3.10) directly determines the functions  $u^{\mu}$  that give the solution  $X$  expressed in (3.9). More generally, the nonholonomic system has solutions if the matrix  $(D_{\mu}^{\alpha})$  has rank  $a_{\circ}$ .

Geometrically, the decomposition  $TN|_M = TM \oplus H$  stated in Corollary 1 has two associated projectors  $\mathcal{P}$ ,  $\mathcal{Q}$ . Writing  $Y = \mathcal{P} \circ Y + \mathcal{Q} \circ Y$  on  $M$ , the following result is clear:

**Proposition 3** *With the preceding notations, if the original system is consistent, with a solution  $Y$ , and the generalized nonholonomic system is regular, with solution  $X$ , the latter can be obtained as*

$$X = \mathcal{P} \circ Y|_M. \quad (3.11)$$

■

Such projectors were studied, in the context of nonholonomic lagrangian systems, in [LM 96].

## 4 Symmetries and constants of motion

Let us consider a generalized nonholonomic system  $(A: TM \rightarrow F, f)$ , obtained from a linearly singular system  $(B: TN \rightarrow G, g)$  by means of a restriction to a submanifold  $M \subset N$  and a projection to the quotient  $p: G|_M \rightarrow (G|_M)/G'$ , where  $G' \subset G|_M$  is a vector subbundle.

Recall the definitions of symmetry and infinitesimal symmetry given in section 2. Our aim is to study the relation between the symmetries of the original linearly singular system on  $N$  and the symmetries of the generalized nonholonomic system on  $M$ . In the next proposition, we give sufficient conditions on a symmetry of the original system in order to define a symmetry of the constrained system:

**Proposition 4** *Let  $(\psi, \Psi)$  be a symmetry of  $(B: TN \rightarrow G, g)$ . Suppose that  $\psi$  leaves the submanifold  $M \subset N$  invariant, and  $\Psi$  leaves the subbundle  $G' \subset G|_M$  invariant. Then  $(\varphi, \Phi)$ , where  $\varphi = \psi|_M$ , and  $\Phi: (G|_M)/G' \rightarrow (G|_M)/G'$  is the map induced on the quotient from  $\Psi$ , is a symmetry of  $(A: TM \rightarrow F, f)$ .*

*Proof.* We have

$$A \circ T\varphi = p \circ B \circ Tj \circ T(\psi|_M) = p \circ B \circ T\psi \circ Tj = p \circ \Psi \circ B \circ Tj = \Phi \circ p \circ B \circ Tj = \Phi \circ A,$$

and

$$f \circ \varphi = p \circ g \circ \psi|_M = p \circ \Psi \circ g|_M = \Phi \circ p \circ g|_M = \Phi \circ f,$$

so the two conditions for being a symmetry are satisfied. ■

We can obtain a similar result for infinitesimal symmetries, by making use of their infinitesimal characterization (2.10):

**Proposition 5** *Let  $(V, \bar{V})$  be an infinitesimal symmetry of  $(B: TN \rightarrow G, g)$ . Suppose that  $V$  is tangent to the submanifold  $M \subset N$ , and  $\bar{V}$  is tangent to the subbundle  $G' \subset G|_M$ . Then  $(U, \bar{U})$ , where  $U = V|_M$  and  $\bar{U}: (G|_M)/G' \rightarrow T((G|_M)/G')$  is the vector field induced on the quotient from  $\bar{V}$ , is an infinitesimal symmetry of  $(A: TM \rightarrow F, f)$ .*

*Proof.* The proof runs as in proposition 4:

$$Tf \circ U = Tp \circ Tg \circ V|_M = Tp \circ \bar{V} \circ g|_M = \bar{U} \circ p \circ g|_M = \bar{U} \circ f,$$

$$\begin{aligned} TA \circ U^T &= Tp \circ TB \circ T(Tj) \circ (V^T)|_{TM} = Tp \circ TB \circ V^T \circ Tj = \\ &= Tp \circ \bar{V} \circ B \circ Tj = \bar{U} \circ p \circ B \circ Tj = \bar{U} \circ A. \end{aligned}$$



■

We now consider constants of motion. Suppose that the original system has a solution  $Y \in \mathfrak{X}(N)$ , and let us consider a function  $h \in C^\infty(N)$  such that  $Y \cdot h = 0$ . Under which conditions is  $h|_M$  a constant of motion of the generalized nonholonomic system?

Suppose that both the original system and the nonholonomic system are regular, so that  $TN|_M = TM \oplus H$ ; let  $\mathcal{P}$  be the projector to the first factor, which, according to Proposition 3, relates the dynamics of both systems as  $X = \mathcal{P} \circ Y$ . Then we have a simple characterization:

**Proposition 6** *With the preceding hypothesis, write  $X = Y - \Gamma$ , where  $\Gamma$  is a section of  $H \subset TN|_M$ . Let  $h$  be a constant of motion of the unconstrained system. Then  $h|_M$  is a constant of motion of the generalized nonholonomic system iff  $\Gamma \cdot h = 0$ .*

*Proof.* It is straightforward:

$$X \cdot h = (Y - \Gamma) \cdot h = Y \cdot h - \Gamma \cdot h.$$

(Note that  $Y$  and  $\Gamma$ , considered as sections of  $TN|_M$ , map functions on  $N$  to functions on  $M$ .) ■

## 5 Lagrangian systems with nonholonomic constraints

In this section we will show that the dynamics of a lagrangian system with nonholonomic constraints (the *nonholonomic mechanics*) falls into the class of generalized nonholonomic systems of section 3.

We begin by considering a configuration manifold  $Q$ , its tangent bundle  $TQ$ , and a lagrangian function  $L: TQ \rightarrow \mathbf{R}$ . The lagrangian mechanics may be described as the linearly singular system  $(\hat{\omega}_L: T(TQ) \rightarrow T^*(TQ), dE_L)$ .

$$\begin{array}{ccc} T(TQ) & \xrightarrow{\hat{\omega}_L} & T^*(TQ) \\ \downarrow & \nearrow dE_L & \\ TQ & & \end{array}$$

Here  $E_L$  is the lagrangian energy and  $\omega_L$  is the Lagrange’s 2-form. Though we do not want to dwell on these well known objects, some properties of  $\omega_L$  and the vertical endomorphism will be needed later, so let us briefly recall them. See [Car 90] for more details.

First, we have the vertical endomorphism  $J$  of  $T(TQ)$ , whose kernel and image are the vertical subbundle  $V(TQ)$ . Its transposed morphism is an endomorphism  ${}^tJ$  of  $T^*(TQ)$ , whose kernel and image are  $Sb(TQ)$ , the bundle of semibasic forms. This is used to define the Lagrange’s 1-form  $\theta_L = {}^tJ \circ dL$  and 2-form  $\omega_L = -d\theta_L$  on  $TQ$ .

From now on we consider the case of the lagrangian being regular, which amounts to  $\omega_L$  being a symplectic form. Then, it induces a vector bundle isomorphism  $\hat{\omega}_L: T(TQ) \rightarrow T^*(TQ)$  mapping vertical vectors to semibasic forms, thus yielding an isomorphism  $V(TQ) \xrightarrow{\cong} Sb(TQ)$ .

It is well known that the lagrangian dynamics on  $TQ$  is described by the only vector field  $X_L$  solution of

$$\hat{\omega}_L \circ X_L = dE_L.$$

Note that it is a second-order vector field.

Now let us introduce the nonholonomic constraints, which define a submanifold  $M \xrightarrow{j} TQ$  of dimension  $m$ . We will consider only the case where this submanifold restricts the velocities, not the configuration coordinates. In a more formal way, this is described by the conditions given in the next proposition:

**Proposition 7** *Let  $M \subset TQ$  be a submanifold. The following conditions are equivalent:*

1. *The projection  $M \rightarrow Q$  (restriction of the tangent bundle projection  $\tau_Q: TQ \rightarrow Q$ ) is a submersion.*
2.  *$(TM)^\perp \cap \text{Sb}(TQ)|_M = 0$ .*
3. *The submanifold  $M \subset TQ$  can be locally described by the vanishing of some constraints  $\phi^i$  whose fibre derivatives  $\mathcal{F}\phi^i$  are linearly independent at each point of  $M$ .*
4. *The submanifold  $M \subset TQ$  can be locally described by the vanishing of some constraints  $\phi^i$  such that the 1-forms  $\Delta^i = {}^tJ \circ d\phi^i$  are linearly independent at each point of  $M$ .*

In coordinates, these conditions mean that  $\left(\frac{\partial \phi^i}{\partial v^k}\right)$  has maximal rank. ■

Note that under the preceding conditions the image  $\tau_Q(M) \subset Q$  is an open submanifold, and so we can replace  $Q$  with this submanifold. So, from now on, we assume that the projection  $M \rightarrow Q$  is a surjective submersion.

Now we will consider the following vector bundles:

$$\begin{aligned} TM &\subset T(TQ)|_M, \\ (TM)^\perp &\subset T^*(TQ)|_M, \\ G' &:= {}^tJ((TM)^\perp) \subset \text{Sb}(TQ)|_M, \\ H &:= \widehat{\omega}^{-1}(G') \subset V(TQ)|_M. \end{aligned}$$

Suppose that  $M \subset TQ$  is defined by the vanishing of some independent constraints  $\phi^i$  as in the preceding proposition. Then  $(TM)^\perp$  is spanned by the  $d\phi^i|_M$ . We denote by  $\Delta^i$  and  $\Gamma_i$  their corresponding images in  $G'$  (through  ${}^tJ$ ) and  $H$  (through  $\widehat{\omega}_L^{-1}$ ). The following diagram shows all these objects:

$$\begin{array}{ccccc} TM \hookrightarrow T(TQ)|_M & \xrightarrow{\widehat{\omega}} & T^*(TQ)|_M & \hookleftarrow & (TM)^\perp = \langle d\phi^i|_M \rangle \\ \updownarrow J & & \updownarrow {}^tJ & & \\ \langle \Gamma_i \rangle = H \hookrightarrow V(TQ)|_M & \xrightarrow{\widehat{\omega}} & \text{Sb}(TQ)|_M & \hookleftarrow & G' = \langle \Delta^i \rangle \end{array}$$

So we have two subbundles  $TM, H \subset T(TQ)|_M$ . We have  $\text{rank } TM = m$  and  $\text{rank}(TM)^\perp = n - m$ ; the conditions in Proposition 7 imply also that  $\text{rank } H = \text{rank } G' = n - m$ .

**Theorem 1** *The nonholonomic mechanics defined by the lagrangian  $L$  and the constraint submanifold  $M \subset TQ$  is the generalized nonholonomic system defined from the lagrangian mechanics  $(\widehat{\omega}_L: T(TQ) \rightarrow T^*(TQ), dE_L)$  by the constraint submanifold  $M \subset TQ$  and the subbundle of constraint forces  $G' = {}^tJ((TM)^\perp) \subset T^*(TQ)|_M$ .*

$$\begin{array}{ccccccc}
 TM & \xrightarrow{\mathring{T}j} & T(TQ)|_M & \xrightarrow{\widehat{\omega}|_M} & T^*(TQ)|_M & \longrightarrow & T^*(TQ)|_M/G' \\
 \downarrow & & \downarrow & \nearrow dE_L|_M & & & \\
 M & \xlongequal{\quad} & M & & & & 
 \end{array}$$

*Proof.* The equation of motion for a path  $\xi = \dot{\gamma}$  such that  $\xi(t) \in M$  is

$$\dot{\xi} = X_L \circ \xi + \sum_i u^i \Gamma_i \circ \xi. \quad (5.1)$$

Instead, let us write the equations of motion for vector fields: according to (3.5), for a second-order vector field  $X$  on  $TQ$ , tangent to  $M$ , the equation is

$$i_X \omega_L \underset{S}{\simeq} dE_L + \sum_i u^i \Delta_i, \quad (5.2)$$

or, according to (3.9),

$$X \underset{S}{\simeq} X_L + \sum_i u^i \Gamma_i. \quad (5.3)$$

But in coordinates equation (5.2) reads

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) = \sum_i u^i \frac{\partial \phi_i}{\partial v},$$

which is the equation of motion of the nonholonomic mechanics defined from  $L$  and the constraints —see for instance [Arn 83]. ■

If, in addition to  $(TM)^\perp \cap \text{Sb}(TQ)|_M = 0$ , we have  $TM \cap H = 0$ , then  $T(TQ)|_M = TM \oplus H$ , and so there is a unique solution  $X$  of the equation of motion, which can be obtained from  $Y$  through the projector to  $TM$  as described by Proposition 3.

### The case of a singular lagrangian

The preceding method can be conveniently adapted if the lagrangian is singular. Of course, one can not use the direct sum decomposition. However, the formulation of the nonholonomic dynamics as a quotient system on a submanifold remains unchanged, except that the second-order condition is not automatically satisfied by  $X$  and must be imposed as an additional equation for it:

$$J \circ X \underset{M}{\simeq} \Delta_{TQ}.$$

This condition may be included in the equation of motion of the nonholonomic dynamics in the same way as can be done with the lagrangian dynamics, using the time-evolution operator

$K$  of lagrangian dynamics [BGPR 86] [GP 89]. With it, the lagrangian dynamics is the linearly singular system

$$\begin{array}{ccc} T(TQ) & \xrightarrow{\mathring{T}(\mathcal{F}L)} & TQ \times_{\mathcal{F}L} T(T^*Q) \\ \downarrow & \nearrow \mathring{K} & \\ TQ & & \end{array}$$

where  $\mathcal{F}L: TQ \rightarrow T^*Q$  is the Legendre’s transformation (fibre derivative) of  $L$ . In terms of vector fields, the lagrangian dynamics is thus defined by the equation

$$T(\mathcal{F}L) \circ X \simeq K.$$

Then, it is readily seen that the nonholonomic equation of motion can be written

$$T(\mathcal{F}L) \circ X \simeq K - \sum_i u^i \Upsilon^{\phi_i}. \quad (5.4)$$

Here  $\Upsilon^\phi$  is a certain vector field along  $\mathcal{F}L$ , which is defined from the fibre derivative of a function  $\phi: T^*Q \rightarrow \mathbf{R}$  —see [GP 01] for details.

## 6 Relativistic particle with a nonholonomic constraint

In this section we study the motion of a relativistic particle as a nonholonomic constrained system. We will consider two possible lagrangian functions, a regular one (deeply studied in [KM01]) and a singular one.

Let us consider a particle with mass  $m$  and charge  $e$  moving in spacetime. We model spacetime as a 4-dimensional manifold  $Q$ , endowed with a metric tensor  $g$  of signature  $(1, 3)$ . Suppose furthermore that the particle is subject to the action of an electromagnetic field  $F = dA$ , where  $A \in \Omega^1(Q)$ , and a potential  $U \in C^\infty(Q)$ .

Recall that there are some relevant objects associated with the metric  $g$ , namely, the isomorphism  $\hat{g}: TQ \rightarrow T^*Q$  (we will denote  $X^\flat = \hat{g} \circ X$ ), the Levi-Civita connection  $\nabla$ , the differential forms  $\theta_g = \hat{g}^*(\theta_Q) \in \Omega^1(TQ)$  and  $\omega_g = \hat{g}^*(\omega_Q) = -d\theta_g \in \Omega^2(TQ)$ , the energy  $E_g(u_q) = \frac{1}{2}g(u_q, u_q) \in C^\infty(TQ)$ , and the geodesic vector field  $S_g$ , which satisfies  $i_{S_g}\omega_g = dE_g$ . We denote  $v = \sqrt{2E_g}$ .

We will study two different lagrangian functions, namely

$$L_1(u_q) = -mc g(u_q, u_q)^{1/2} - \frac{e}{c} \langle A(q), u_q \rangle - U(q),$$

and

$$L_2(u_q) = -\frac{1}{2}m g(u_q, u_q) - \frac{e}{c} \langle A(q), u_q \rangle - U(q).$$

Forgetting the potential,  $L_1$  is the singular lagrangian commonly used in relativistic mechanics to describe a particle in an electromagnetic field; it is defined only on the open set of time-like vectors of  $TQ$ . The lagrangian  $L_2$  appears in [KM01]. Our aim is to compare both systems, and to introduce the nonholonomic constraint  $v^2 = c^2$  to them.

The lagrangians  $L_1$  and  $L_2$  have, respectively, associated Lagrange’s 1-forms  $\theta_1 = -\frac{mc}{v}\theta_g - \frac{e}{c}\tau_Q^*A$  and  $\theta_2 = -m\theta_g - \frac{e}{c}\tau_Q^*A$ ; the Lagrange’s 2-forms are  $\omega_1 = -\frac{mc}{v}\omega_g - \frac{e}{v^2}dv \wedge \theta_g + \frac{e}{c}\tau_Q^*F$  and  $\omega_2 = -m\omega_g + \frac{e}{c}\tau_Q^*F$ ; and the lagrangian energies are  $E_1 = U$  and  $E_2 = -\frac{1}{2}mv^2 + U$ .

The symplectic formulation of the equations of motion for the lagrangians  $L_1$  and  $L_2$  are, respectively,

$$i_X\omega_1 = dE_1, \quad (6.1)$$

and

$$i_X\omega_2 = dE_2, \quad (6.2)$$

for second-order vector fields  $X$ . For any 2-form  $\omega$ , we will also denote  $i_X\omega$  by  $\hat{\omega}(X)$ .

It is worth writing down the Euler–Lagrange equations of motion for a path  $\gamma$ , which are, for lagrangians  $L_1$  and  $L_2$ :

$$\frac{mc}{g(\dot{\gamma}, \dot{\gamma})^{1/2}} \left( (\nabla_t \dot{\gamma})^b - \frac{g(\dot{\gamma}, \nabla_t \dot{\gamma})}{g(\dot{\gamma}, \dot{\gamma})} \dot{\gamma}^b \right) + \frac{e}{c} i_{\dot{\gamma}} F - dU = 0, \quad (6.3)$$

and

$$m(\nabla_t \dot{\gamma})^b + \frac{e}{c} i_{\dot{\gamma}} F - dU = 0. \quad (6.4)$$

Let us now consider equations (6.1) and (6.2).

As  $\hat{\omega}_1$  is not surjective, equation (6.1) could have no solutions. We denote by  $\Delta = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$  the Liouville vector field,  $T = \dot{q}^i \frac{\partial}{\partial \dot{q}^i}$  the natural vector field along  $\tau_Q$ , and  $\xi^\vee$  the vertical lift of a vector field  $\xi: TQ \rightarrow TQ$  along  $\tau_Q$ . We have that  $\text{Ker } \omega_1 = \langle \Delta, \Sigma \rangle$ , where

$$\Sigma = S_g - \frac{ev}{mc^2} ((i_T F)^\sharp)^\vee. \quad (6.5)$$

We can see that  $\hat{\omega}_1(\frac{v}{mc}(\text{grad } U)^\vee) = dU - (\frac{1}{v^2} i_T dU) \theta_g$  and that  $\theta_g \notin \text{Im } \hat{\omega}_1$ . Therefore equation (6.1) has solutions if and only if  $i_T dU = 0$ , that is, the potential  $U$  is constant, which, in practice, is the same as taking  $U$  equal to 0.

Since  $\Sigma$  is a second-order vector field, in absence of potential the solutions of equation (6.1) are  $X_1 = \Sigma + \mu\Delta$ , where  $\mu$  is an arbitrary function. If, in addition, there is no electromagnetic field, then the solutions are  $S_g + \mu\Delta$ , and their integral curves are reparametrized geodesics.

On the other hand, equation (6.2) is regular, and its solution is

$$X_2 = S_g + \frac{1}{m}(\text{grad } U)^\vee - \frac{e}{mc} ((i_T F)^\sharp)^\vee. \quad (6.6)$$

This can be proved making use of the relations  $i_{Z^\vee} \omega_g = -\tau_Q^*(Z^b)$  for vector fields  $Z$  along  $\tau_Q$ , and  $i_S(\tau_Q^* F) = \tau_Q^*(i_T F)$ . In this case, in absence of electromagnetic field and potential, the solutions are the geodesics of  $g$ .

Now we introduce the nonholonomic constraint

$$\phi(u_q) := g(u_q, u_q) - c^2 = 0, \quad (6.7)$$

which defines a submanifold  $M \subset TQ$ .

The subbundle of constraint forces is  $\langle {}^tJ(d\phi) \rangle|_M = \langle \theta_g \rangle|_M$ , therefore, according to equation (5.2), the equations of motion for both lagrangians become

$$i_X \omega_1 \underset{M}{\simeq} dE_1 + \lambda \theta_g, \quad (6.8)$$

and

$$i_X \omega_2 \underset{M}{\simeq} dE_2 + \lambda \theta_g, \quad (6.9)$$

for second-order vector fields  $X$  tangent to  $M$ .

Note that if a path  $\gamma$  satisfies the constraint then it also satisfies the equation  $0 = \frac{d}{dt}g(\dot{\gamma}, \dot{\gamma}) = 2g(\dot{\gamma}, \nabla_t \dot{\gamma})$ , so looking at equations (6.3) and (6.4) we realize that the two *constrained* systems have the *same* equations of motion:

$$\begin{cases} m(\nabla_t \dot{\gamma})^b + \frac{e}{c} i_{\dot{\gamma}} F - dU = \lambda \dot{\gamma}^b, \\ g(\dot{\gamma}, \dot{\gamma}) = c^2. \end{cases} \quad (6.10)$$

The multiplier  $\lambda$  can be found by contracting the equation with  $\dot{\gamma}$ , which gives  $\lambda = -\frac{1}{c^2} i_{\dot{\gamma}} dU$ .

We are going to see this equivalence of the solutions of both Euler–Lagrange equations by computing the solutions of equations (6.8) and (6.9).

First let us analyse equation (6.9). From  $\Delta \cdot \phi = 2v^2 \underset{M}{\simeq} 2c^2 \neq 0$  and  $i_\Delta \omega_2 = m\theta_g$ , it follows that  $TM \oplus \widehat{\omega}_2^{-1}(\langle \theta_g \rangle|_M) = (TQ)|_M$ , so, by proposition 1, the system is regular. Its solution is  $X = X_2 + \frac{\lambda}{m} \Delta$ , where the multiplier  $\lambda$  is found by imposing that  $X$  is tangent to  $M$ :

$$0 = X \cdot \phi = X_2 \cdot \phi + \frac{\lambda}{m} \Delta \cdot \phi \underset{M}{\simeq} \frac{2}{m} i_T dU + 2 \frac{\lambda}{m} c^2. \quad (6.11)$$

Therefore, the solution of the second system is

$$X = S_g + \frac{1}{m} (\text{grad } U)^\vee - \frac{e}{mc} ((i_T F)^\#)^\vee - \frac{1}{mc^2} (i_T dU) \Delta. \quad (6.12)$$

Now let us analyse equation (6.8). Since  $Y = \frac{1}{m} (\text{grad } U)^\vee - \frac{1}{mc^2} (i_T dU) \Delta$  is a vector field tangent to  $M$  and  $\widehat{\omega}_1(Y) \underset{M}{\simeq} dU - (\frac{1}{c^2} i_T dU) \theta_g$ , the system is consistent. We can see that

$$TM \cap \widehat{\omega}_1^{-1}(\langle \theta_g \rangle|_M) = TM \cap \text{Ker } \widehat{\omega}_1 = \langle \Sigma \rangle|_M, \quad (6.13)$$

so the system is not regular. Then, the solutions of the equation are  $Y + \mu \Sigma$ . Since  $Y$  is vertical, in order to be a second-order vector field the function  $\mu$  must be equal to one, so the solution is  $Y + \Sigma \underset{M}{\simeq} X$ , exactly the same as for the lagrangian  $L_2$ .

## 7 Examples

### Example 1

Consider the differential equation on  $N = \mathbf{R}^2$  defined by the vector field  $Y = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ . We restrict this system to a generalized nonholonomic one by means of the construction of section 3, taking the submanifold  $M = \mathbf{R} \times \{a\} \subset N$  and the subbundle  $C = \langle x \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \rangle \subset \text{TN}|_M$ .

In this case  $TN|_M = TM \oplus C$  and the projectors associated with this decomposition are

$$\begin{aligned} \mathcal{P}: \quad \frac{\partial}{\partial x} &\longmapsto \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} &\longmapsto -x \frac{\partial}{\partial x}, \\ \mathcal{Q}: \quad \frac{\partial}{\partial x} &\longmapsto 0 \\ \frac{\partial}{\partial y} &\longmapsto x \frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \end{aligned}$$

Thus  $X = \mathcal{P} \circ Y|_M = (1 - ax) \frac{\partial}{\partial x}|_M$  is the solution of the generalized nonholonomic system.

Let us study the infinitesimal symmetries of both systems. We can see that a vector field  $V \in \mathcal{X}(N)$  is an infinitesimal symmetry of the unconstrained system if it has the form  $V = V^1(ye^{-x}) \frac{\partial}{\partial x} + e^x V^2(ye^{-x}) \frac{\partial}{\partial y}$ , where  $V^1$  and  $V^2$  are arbitrary smooth functions.

On the other hand, since the constrained system is one-dimensional, its infinitesimal symmetries are the vector fields  $U = kX$ , with  $k \in \mathbf{R}$ . Observe that, in principle, an infinitesimal symmetry of  $Y$  does not lead to an infinitesimal symmetry of  $X$  by restriction to  $M$ , even when  $Y|_M \in \mathfrak{X}(M)$ . Nevertheless, if we also require that  $V^T(C) \subset TC$ , then we obtain  $V^1(t) = k(1 + a \ln(t/a))$  and  $V^2(t) = 0$ , so that actually  $V|_M = k(1 - ax) \frac{\partial}{\partial x}|_M$  is an infinitesimal symmetry of  $X$ .

## Example 2

Here we discuss an example of a particle with a nonholonomic constraint, due to Rosenberg [Ros 77]. Consider a particle moving in  $\mathbf{R}^3$  with lagrangian function

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

subject to the nonholonomic constraint

$$\phi = \dot{z} - y\dot{x}.$$

Using the notation of section 5, we have  $N = \mathbf{TR}^3$ ,  $\omega_L = dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}$  and  $dE_L = \dot{x}d\dot{x} + \dot{y}d\dot{y} + \dot{z}d\dot{z}$ , so the unconstrained dynamics is the well-known free dynamics described by the vector field

$$X_L = \widehat{\omega}_L^{-1}(dE_L) = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z}.$$

The constraint submanifold is  $M = \{\dot{z} = y\dot{x}\}$ , with tangent bundle

$$TM = \text{Ker}(d\phi) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + \dot{x} \frac{\partial}{\partial \dot{z}}, \frac{\partial}{\partial \dot{x}}, \frac{\partial}{\partial \dot{y}} + y \frac{\partial}{\partial \dot{z}}, \frac{\partial}{\partial \dot{z}} \right\rangle \Big|_M,$$

and the vector subbundle  $C \subset TN|_M$  is

$$C = \langle \widehat{\omega}_L^{-1}(^tJ(d\phi)) \rangle = \left\langle y \frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial \dot{z}} \right\rangle \Big|_M.$$

Note that  $TN|_M$  splits as  $TN|_M = TM \oplus C$ , so the only solution  $X$  of the constrained lagrangian system is the projection of  $X_L|_M$  to  $TM$  according to this decomposition:

$$X = \left( \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \dot{z} \frac{\partial}{\partial z} - \frac{y\dot{y}\dot{x}}{y^2 + 1} \frac{\partial}{\partial \dot{x}} + \frac{y\dot{x}}{y^2 + 1} \frac{\partial}{\partial \dot{z}} \right) \Big|_M.$$

We choose  $(x, y, z, \dot{x}, \dot{y})$  as coordinates on  $M$ . With this system, the vector field  $X$  reads as

$$X = \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + y \dot{x} \frac{\partial}{\partial z} - \frac{y \dot{y} \dot{x}}{y^2 + 1} \frac{\partial}{\partial \dot{x}}.$$

After some calculus, we can find the symmetries and constants of motion of both systems. The constants of motion of the free particle are the functions  $G(\dot{x}, \dot{y}, \dot{z}, \dot{x}y - \dot{y}x, \dot{y}z - \dot{z}y)$ , where  $G$  is an arbitrary function with five variables. The infinitesimal symmetries are linear combinations of the six vector fields  $\frac{\partial}{\partial x}$ ,  $\frac{\partial}{\partial y}$ ,  $\frac{\partial}{\partial z}$ ,  $x \frac{\partial}{\partial x} + \dot{x} \frac{\partial}{\partial \dot{x}}$ ,  $y \frac{\partial}{\partial y} + \dot{y} \frac{\partial}{\partial \dot{y}}$  and  $z \frac{\partial}{\partial z} + \dot{z} \frac{\partial}{\partial \dot{z}}$ , with the constants of motion as coefficients.

The constants of motion of the constrained system, written in coordinates of  $M$ , are

$$F\left(\dot{y}, \dot{x}\sqrt{y^2 + 1}, \dot{y}x - \operatorname{arcsinh}(y)\dot{x}\sqrt{y^2 + 1}, \dot{y}z - \dot{x}(y^2 + 1)\right), \quad (7.1)$$

and the infinitesimal symmetries are linear combinations of the five vector fields

$$\begin{aligned} & \frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + y \dot{x} \frac{\partial}{\partial z} - \frac{\dot{x} \dot{y} y}{y^2 + 1} \frac{\partial}{\partial \dot{x}}, \\ & \frac{\operatorname{argsinh}(y)}{\dot{y}} \frac{\partial}{\partial x} + \frac{\sqrt{y^2 + 1}}{\dot{y}} \frac{\partial}{\partial z} + \frac{1}{\sqrt{y^2 + 1}} \frac{\partial}{\partial \dot{x}}, \\ & \frac{\dot{x}(y - \operatorname{argsinh}(y)\sqrt{y^2 + 1})}{\dot{y}^2} \frac{\partial}{\partial x} + \frac{y}{\dot{y}} \frac{\partial}{\partial y} - \frac{\dot{x}}{\dot{y}^2} \frac{\partial}{\partial z} - \frac{\dot{x} y^2}{\dot{y}(y^2 + 1)} \frac{\partial}{\partial \dot{x}} + \frac{\partial}{\partial \dot{y}}, \end{aligned}$$

with the constants of motion as coefficients.

In order to illustrate proposition 6 we take a function  $g = G(\dot{x}, \dot{y}, \dot{z}, \dot{x}y - \dot{y}x, \dot{y}z - \dot{z}y)$ , i.e., a constant of motion of  $X_L$ , such that  $Z \cdot g = 0$ , where  $Z$  is the section of  $C$

$$Z = X_L|_M - X = \frac{\dot{x} \dot{y}}{y^2 + 1} \left( y \frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial \dot{z}} \right) \Big|_M.$$

This yields to

$$g = H\left(\dot{y}, \sqrt{\dot{z}^2 + \dot{x}^2}, \dot{z} + \dot{y}x - \dot{x}y - \operatorname{argsinh}(\dot{z}/\dot{x})\sqrt{\dot{z}^2 + \dot{x}^2}, \dot{y}z - \dot{z}y - \dot{x}\right)$$

and we see that  $g|_M$  is just the expression (7.1).

## Appendix: some lemmas about linear algebra

Here we collect some results about linear algebra on vector bundles that are needed in section 3. These lemmas are stated and proved for vector spaces, but of course nothing changes essentially if vector bundles are considered instead.

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ j \uparrow & & \downarrow p \\ E_o & \xrightarrow{f} & F/F_o \end{array}$$



**Lemma 1** *Let  $f: E \rightarrow F$  be a linear map between vector spaces, and  $E_o \subset E$  and  $F_o \subset F$  vector subspaces. Denote  $j: E_o \rightarrow E$  the inclusion,  $p: F \rightarrow F/F_o$  the projection to the quotient, and consider the composition  $\bar{f} = p \circ f \circ j$ . Then:*

1.  $\bar{f}$  is injective iff  $E_o \cap f^{-1}(F_o) = \{0\}$ .  
Assuming  $f$  injective, this also amounts to  $f(E_o) \cap F_o = \{0\}$ .
2.  $\bar{f}$  is surjective iff  $f(E_o) + F_o = F$ .  
Assuming  $f$  surjective, this also amounts to  $E_o + f^{-1}(F_o) = E$ .
3. When  $f$  is surjective,  $\bar{f}$  is bijective iff  $E_o \oplus f^{-1}(F_o) = E$ .  
When  $f$  is injective,  $\bar{f}$  is bijective iff  $f(E_o) \oplus F_o = F$ .

*Proof.* First note that

$$\text{Ker } \bar{f} = E_o \cap f^{-1}(F_o), \quad \text{Im } \bar{f} = (f(E_o) + F_o)/F_o. \quad (\text{A.1})$$

These equalities are clear: the kernel is constituted by the vectors in  $E_o$  mapped to  $F_o$  by  $f$ , and the image of a subspace  $F' \subset F$  by  $p$  is  $(F' + F_o)/F_o$ . This readily yields the first assertions about injectivity and surjectivity.

Their equivalent formulations when  $f$  is injective [or surjective] can be proved using the formulas for  $f(E_1 \cap E_2)$  and  $f^{-1}(F_1 \cap F_2)$  [or for the sum], as well as  $f^{-1}(f(E_o)) = E_o + \text{Ker } f$ ,  $f(f^{-1}(F_o)) = F_o \cap \text{Im } f$ .

Finally, the assertions about the bijectivity of  $\bar{f}$  are a trivial consequence of the other ones. ■

Remember that a linear equation  $f(x) = b$  is consistent iff  $b \in \text{Im } f$ . Now let us study a linear equation on  $E_o$  defined as in the preceding lemma by  $\bar{f}$  and the class of an element  $b \in F$ .

**Lemma 2** *The linear equation  $\bar{f}(x) = \bar{b}$  is equivalent to the couple of equations  $f(x) - b \in F_o$ ,  $x \in E_o$ . It is consistent iff  $b \in f(E_o) + F_o$ ; in this case the solution is unique iff  $E_o \cap f^{-1}(F_o) = \{0\}$ . ■*

Finally, let  $E \subset G$  be a subspace of a vector space. Recall that the *annihilator* (or orthogonal) of  $E$  is the subspace

$$E^\perp = \{\gamma \in G^* \mid (\forall x \in E) \langle \gamma, x \rangle = 0\} \subset G^*.$$

This space has a close relationship with  $G/E$ . Indeed, the transpose map of  $G \rightarrow G/E$  defines a canonical isomorphism

$$\delta: (G/E)^* \rightarrow E^\perp,$$

such that, for  $\alpha \in E^\perp$  and  $z \in G$ ,  $\langle \delta^{-1}(\alpha), z + E \rangle = \langle \alpha, z \rangle$ .

**Lemma 3** *Let  $E, F \subset G$  be vector subspaces. Let  $(\alpha^1, \dots, \alpha^p)$  be a basis for the annihilator  $E^\perp \subset G^*$ , and  $(v_1, \dots, v_q)$  a basis for  $F$ . Consider the matrix  $D = (D_j^i)_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}}$  with elements  $D_j^i = \langle \alpha^i, v_j \rangle$ . Then:*

1.  $E + F = G$  iff  $\text{rank } D = p$ .

2.  $E \cap F = \{0\}$  iff  $\text{rank } D = q$ .
3.  $E \oplus F = G$  iff  $D$  is square invertible.

*Proof.* Consider the linear map  $\varepsilon: F \rightarrow G/E$  defined as the composition of the inclusion  $F \hookrightarrow G$  and the projection to the quotient  $G \twoheadrightarrow G/E$ . It is clear that  $E + F = G$  iff  $\varepsilon$  is surjective, and  $E \cap F = \{0\}$  iff  $\varepsilon$  is injective, so the only thing to prove is that the given matrix is the matrix  $D$  of  $\varepsilon$  in appropriate bases: the basis  $(v_j)$  for  $F$ , and the basis  $(\bar{\alpha}_i)$ , the dual basis of  $\bar{\alpha}^i = \delta^{-1}(\alpha^i)$ , for  $G/E$ .

Then, if  $\varepsilon(v_j) = \bar{\alpha}_i D_j^i$ , we have  $D_j^i = \langle \bar{\alpha}^i, \varepsilon(v_j) \rangle = \langle \alpha^i, v_j \rangle$ , which is what we wanted to prove. ■

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